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A METHOD OF DETERMINING
A CONFIDENCE INTERVAL FOR AVAILABILITY

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THIS REPORT HAS BEEN PREPARED PRIMARILY FOR TIMELY PRESENTATION OF INFORMATION. ALTHOUGH CARE HAS BEEN TAKEN IN THE PREPARATION OF THE TECHNICAL MATERIAL PRESENTED, CONCLUSIONS DRAWN ARE NOT NECESSARILY FINAL AND MAY BE SUBJECT TO REVISION.

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SUMMARY

This report presents a method of determining a confidence interval for availability when it is estimated from the mean time between equipment failures and the mean time to repair these failures.

When the failure rate and repair rate of a system are constants independent of time, it can be shown that the estimates of mean time between failures and of mean time to repair the failures can be combined to have the variance ratio density. The estimated availability can be shown to have a density which is a function of the variance ratio density. This relation can then be used to obtain equations for a confidence interval for availability.

An example is included to demonstrate the procedure of placing a confidence interval about the estimated availability.

INTRODUCTION

At the U. S. Naval Missile Center (NMC), the term "availability" has been used as a measure of the probability that a weapon system will be ready at a prescribed time to perform its intended function. In this sense, availability (A) is defined as

$$A = \frac{\theta}{\theta + \phi}$$

where θ is the mean time between equipment failures and ϕ is the mean time to repair these failures. The actual values of these parameters, though unknown, are represented by the estimates $\hat{\theta}$ and $\hat{\phi}$. Since $\hat{\theta}$ and $\hat{\phi}$ are calculated from data, they are subject to random variation inherent in sampling. The estimate of availability

$$\hat{A} = \frac{\hat{\theta}}{\hat{\theta} + \hat{\phi}}$$

is then also subject to random variation and can be expected to change from sample to sample even though the actual value of availability is constant over these samples.

In the past, availability has been reported as a point estimate, or single value. A point estimate of a parameter is not very meaningful without some measure of the possible error in the estimate. An estimate of a parameter should be accompanied by some interval, or range of values, about the estimate together with some measure of assurance that the actual value of the parameter does lie within the interval. This interval is commonly known as a confidence interval.

A literature search did not disclose any documented method for determining a confidence interval for availability. The purpose of this report is to document such a method. In order to calculate a confidence interval for availability, it is first necessary to determine the probability density of some function of the estimate $\hat{A}$. This report is specifically concerned with deriving a a probability density function under some limiting assumptions and demonstrating how it may be used to calculate the limits of a confidence interval about $\hat{A}$.

DERIVATION OF THE DENSITY FUNCTION OF $\hat{\theta}$

If it is assumed that the failure rate of a system, $\frac{1}{\theta}$, is a constant independent of time, then the density function of t_i (the operating time between the (i-1)th and the ith failure) is

$$f(t_i) = \frac{1}{\theta} \exp\left(-\frac{t_i}{\theta}\right)$$

By definition

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} t_i$$

where n is the number of time periods.

The joint density function of $(t_1, t_2, \ldots, t_n)$ is

$$f(t_1, t_2, \ldots, t_n) = \left(\frac{1}{\theta}\right)^n \exp\left(\sum_{i=1}^{n} - \frac{t_i}{\theta}\right)$$

By transformation, the joint density function of $(\hat{\theta}, t_2, t_3, \ldots, t_n)$ is obtained:

$$g(\hat{\theta}, t_2, t_3, \ldots, t_n) = n\left(\frac{1}{\theta}\right)^n \exp\left(-\frac{n\hat{\theta}}{\theta}\right)$$

The marginal density function, $g(\hat{\theta})$, is obtained by integrating over the variables $t_2, t_3, \ldots, t_n$:

$$g(\hat{\theta}) = \frac{\left(\frac{1}{\theta}\right)^n \hat{\theta}^{\,n-1} n^n \exp\left(-\frac{n\hat{\theta}}{\theta}\right)}{(n-1)!}$$

Let the random variable $u = \dfrac{2n\hat{\theta}}{\theta}$. By a change of variable, the density function of $g(u)$ is obtained:

$$g(u) = \frac{u^{n-1}\exp\left(-\frac{u}{2}\right)}{2^n\,(n-1)!}$$

which is a chi-squared (χ^2) density function with 2n degrees of freedom.

DERIVATION OF THE DENSITY FUNCTION OF $\hat{\phi}$

If it is assumed that the repair rate of a system, $\dfrac{1}{\phi}$, is a constant independent of time, then the density function of t_j (the time required to repair the jth failure) is

$$f(t_j) = \frac{1}{\phi}\exp\left(-\frac{t_j}{\phi}\right)$$

By definition

$$\hat{\phi} = \frac{1}{m}\sum_{j=1}^{m} t_j$$

where m is the number of repairs.

The density function of $\hat{\phi}$ can be obtained by a procedure similar to the one used for the density function of $\hat{\theta}$. Thus

$$g(\hat{\phi}) = \frac{\left(\frac{1}{\phi}\right)^m \hat{\phi}^{m-1} \; m^m \exp\left(-\frac{m\hat{\phi}}{\phi}\right)}{(m-1)!}$$

Let the random variable $v = \frac{2m\hat{\phi}}{\phi}$. By a change of variable, the density function of g(v) is obtained:

$$g(v) = \frac{v^{m-1} \exp\left(-\frac{v}{2}\right)}{2^m (m-1)!}$$

which is a χ^2 density function with 2m degrees of freedom.

DEGREES OF FREEDOM

In the derivation of the density function of the estimate $\hat{\theta}$, it was assumed that each operating time t_i terminated in a failure. Therefore, there were n values of t_i and n failures. The random variable $u = \frac{2n\hat{\theta}}{\theta}$ is distributed as χ^2 with 2n degrees of freedom.

In the event that the last operating time, t_n, does not terminate in a failure but terminates for some other reason, such as the end of a monitoring period, and $\hat{\theta}$ is calculated as the total operating time divided by the number of failures, (n-1), then the quantity $u = \frac{2(n-1)\hat{\theta}}{\theta}$ is distributed as χ^2 with 2n degrees of freedom.

For the first case, the number of degrees of freedom associated with the random variable u is equal to twice the number of failures. For the second case, the number of degrees of freedom associated with the random variable u is equal to twice the number of failures plus two.

In most situations, such as the life testing of equipment, it is readily apparent which one of the two cases applies. If the test is terminated at a predetermined number of failures, then the first case applies. If the test is terminated at a predetermined operating time, then the second case applies.

In the situation where monitoring of systems is terminated at a predetermined calendar time (for instance, one week), then a decision must be made as to which of the two cases applies. Since a failure occurs at a discrete point in time, it is highly unlikely that a monitoring period terminating at the end of a period of calendar time would also terminate exactly at a failure. By this argument, the second case would apply.

In the derivation of the density function of the estimate $\hat{\phi}$, it was assumed that each repair time t_j terminated in a repair. Therefore, the random variable $v = \frac{2m\hat{\phi}}{\phi}$ is distributed as χ^2 with 2m degrees of freedom.

CONFIDENCE INTERVAL ABOUT $\hat{A}$

Since u and v are independently distributed as χ^2 with 2n and 2m degrees of freedom, the quantity

$$\frac{u/2n}{v/2m}$$

has the variance ratio density (F) with 2n and 2m degrees of freedom.

The following relations can be used to obtain the upper (U) and lower (L) confidence limits of a confidence interval about $\frac{\hat{\phi}}{\hat{\theta}}$.

CASE 1

$$\left(\frac{\phi}{\theta}\right)_U = \frac{\hat{\phi}}{\hat{\theta}} F_{\frac{1-\alpha}{2};\,2n,\,2m}$$

$$\left(\frac{\phi}{\theta}\right)_L = \frac{\hat{\phi}}{\hat{\theta}} F_{\frac{\alpha}{2};\,2n,\,2m}$$

CASE 2

$$\left(\frac{\phi}{\theta}\right)_U = \frac{n}{n-1}\,\frac{\hat{\phi}}{\hat{\theta}} F_{\frac{1-\alpha}{2};\,2n,\,2m}$$

$$\left(\frac{\phi}{\theta}\right)_L = \frac{n}{n-1}\,\frac{\hat{\phi}}{\hat{\theta}} F_{\frac{\alpha}{2};\,2n,\,2m}$$

where

n = the number of failures in case 1

n = the number of failures plus one in case 2

m - the number of repairs

$1-\alpha$ = the level of confidence

$F_{\frac{1-\alpha}{2};\,2n,\,2m}$ - the upper $\frac{1-\alpha}{2}$ fractile of the cumulative F density with 2n and 2m degrees of freedom

$F_{\frac{\alpha}{2};\,2n,\,2m}$ - the lower $\frac{\alpha}{2}$ fractile of the cumulative F density with 2n and 2m degrees of freedom

If availability is expressed as

$$A = \frac{1}{1 + \left(\frac{\phi}{\theta}\right)}$$

then the upper and lower limits of a confidence interval about $\hat{A}$ may be obtained from the following relations:

$$A_U = \frac{1}{1 + \left(\frac{\phi}{\theta}\right)_L}$$

$$A_L = \frac{1}{1 + \left(\frac{\phi}{\theta}\right)_U}$$

EXAMPLE OF DETERMINATION OF A CONFIDENCE INTERVAL

A typical situation at NMC may involve the monitoring of an F-4B squadron during a one-week weapons training exercise. The following is an example of how a confidence interval may be determined from data collected during one of these exercises. Given the following fictitious data:

Total equipment "on" time . . . . .200 hours
Total number of failures 14
Total time expended in repair. . . . . .100 hours
Total number of repairs. 20*

then

$$\hat{\theta} = \frac{200}{14} = 14.286$$

$$\hat{\phi} = \frac{100}{20} = 5.000$$

$$\hat{A} = \frac{\hat{\theta}}{\hat{\theta} + \hat{\phi}} = 0.741$$

For a 95 per cent confidence interval on availability

$$\left(\frac{\phi}{\theta}\right)_U = \frac{n}{n-1} \frac{\hat{\phi}}{\hat{\theta}} F_{0.975\ (30,\ 40)}$$

$$= \left(\frac{15}{14}\right)\left(\frac{5.000}{14.286}\right)(1.94) = 0.7275$$

*It is possible for there to be more repairs than failures if several aircraft begin the exercise with failures.

$$\left(\frac{\phi}{\theta}\right)_L = \frac{n}{n-1} \quad \frac{\hat{\phi}}{\hat{\theta}} \, F_{0.025\,(30,\,40)}$$

$$= \left(\frac{15}{14}\right)\left(\frac{5.000}{14.286}\right)\left(\frac{1}{2.01}\right) = 0.1866$$

$$A_U = \frac{1}{1 + \left(\frac{\phi}{\theta}\right)_L} = \frac{1}{1.1866} = 0.8428$$

$$A_L = \frac{1}{1 + \left(\frac{\phi}{\theta}\right)_U} = \frac{1}{1.7275} = 0.5789$$